**Conductance near Anderson Transition**

**Paper**

So we are characterizing the system by two parameters LT/ξ and Lz/ξ. And he says that we’re going to obtain differential equations for the conductance distribution P(g) for any D, that reduce to Q1D in the appropriate limit, and that reproduce the g-cumulants obtained by the NLσM in D = 2 + ε.

**1. 3D behavior in all regimes**

Further he says that he will find the distribution is log-normal for super small g, and exponential(ly damped) for super large g, in all regimes.

**1.a 3D behavior in insulating regime**

The prior statement would suggest that the insulating regime distribution is therefore effectively log-normal, and he claims as much (cites a 2001 paper though, and i thought consensus was that distribution was not log-normal except in Q1D). Here’s a reminder of what log-normal is:



**1.b 3D behavior in conducting regime**

And metallic regime is Gaussian, with the log-normal/exponential asymptotics confined to the nether regions of the distribution.

Goes on to survey what’s known numerically about the distribution: Gaussian in metallic regime, and log-normal in insulating regime (?).

**1.c 3D behavior in critical regime**

Critical regime is a combination of the two: log-normal on left and exponential on the right. Mentions the singularity thought to be present around g = 1 (I was never able to reproduce it) and expresses skepticism based on singularities only being present in the thermodynamic limit. I imagine that’s no singularities for thermodynamic functions and their derivatives, at certain critical temperatures. Why should that pertain here?

**D = 2+ε behavior in critical regime & NLσM discussion of ambiguity**

Makes a remark about validity of single parameter scaling theory – can prove by graphing <g>n vs. L/ξ for instance. All cumulants should only depend on that parameter. So according to the conventional interpretation of the single-parameter scaling hypothesis, the conductance distribution, entire, should be length-invariant at the critical point. But the NLσM produced the following results for the g-cumulants in D = 2+ε, at the critical point.



**D = 2:**

[I’m guessing 2/ε is the tipping point, because otherwise some moments will vanish with L and some will diverge, which wouldn’t make sense] So note that all cumulants are equal and length-independent in D = 2 (and seems to suggest the average diverges? but can’t, i guess n ≥ 2 here implicitly).

**2 < D < 3:**

As D grows between 2 and 3, more and more (lower) cumulants become length-dependent (and in fact diverge in the L = ∞ limit).

**D = 3:**

In D = 3, only the first two cumulants (average, and variance) are length-independent. And the rest have diverged to ∞.

**Interpretation vis a vis supposed P(g) invarance at critical point**

So one can suppose that either the single-parameter scaling theory is invalid. Or, one could propose that the NLσM gets progressively worse as D → 3, since more and more cumulants of the distribution become length-dependent (and ultimately diverge in the L → ∞ limit). But a compromise position is taken up by Shapiro who takes the results at face value. He suggests that the single-parameter scaling theory is valid in the main, i.e., for <g> and maybe var(g), so that the bulk of the distribution does obey the scaling hypothesis, but as the NLσM suggests, its more or less negligible tails may not (and observe that it’s the tails of the distribution that tend to carry heavier weight vis a vis the large cumulants since <g>n gets larger with |g|). To wit, he used a Migdal-Kadanoff scaling approach to get the probability for ρ = 1/g, at the critical point in D = 2+ε (this wouldn’t seem to converge at origin for ε = 1):



this would be equivalent to:



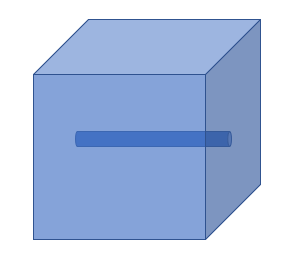
hmmmm (and I’m guessing this is for L = ∞ ‘cause otherwise there should be some length dependence accoding to his hypothesis). Anyway, Suslov points out, I think, that if we got Pc(g) ~ 1/(1+g)4/ε or something (guessing for L = ∞ again), at the critical point, this would be consistent with the NLσM results above in that, roughly, cumulants <g>n would converge for n < 2/ε, and be *infinite* otherwise (which would suggest that at finite L these infinite moments would be large and porportional to L like the NLσM asserts). But then he points out that, from numerical experiments, P(g) does not appear to have power law tails at D = 3 (ε = 1). I guess Suslov is going to accept Shapiro’s compromise position on the single-parameter scaling theory. Rather, they appear to be exponentially damped. So what I think he is saying is that P(g) is always exponentially damped for finite lengths, but in the infinite length limit, there remains power law tails, something like:



or so. Still have to reconcile this form with the 3D critical regime ln-normal-exponential distributiony thing.

**2. Basic Shapiro approach**

Starts off with Shapiro’s basic approach, which he will later modify. So we start with a 1D wire, embedded in a dielectric cube. Then we stack cubes to build larger cubes and see how the conductances scales.



Basically, it appears that we assume the resistance scales quantumly in the longitudinal direction, and classically in the transverse direction. The probability distribution of resistance, ρ = 1+λ = 1/g of one of these chains is given by:



**2.1 Scaling of <G> assuming identical chains: his approach**

Now suppose we know the conductance of one of these chains. Then we will know the conductance of the entire chain, because, assuming all chains are identical, we can write:



And we know, from the 1D equation, that:



where <g> = <T> = <1/(1+λ)> = <1/ρ>. Seems that the scaling law may not require a particular initial shape per seʹ, but just that we scale all dimensions by the same factor. Well I’d be suspicious here, because a massively Q1D wire would probably always scale towards insulator regardless of disorder – at least numerical simulations seems to suggest as much. For a wire of length bz, we have:



and then if we add bd-1 wires to the side of it, this should decrease the resistance by the same factor



Now it seems we just forget that our building block is a 1D wire, and write:



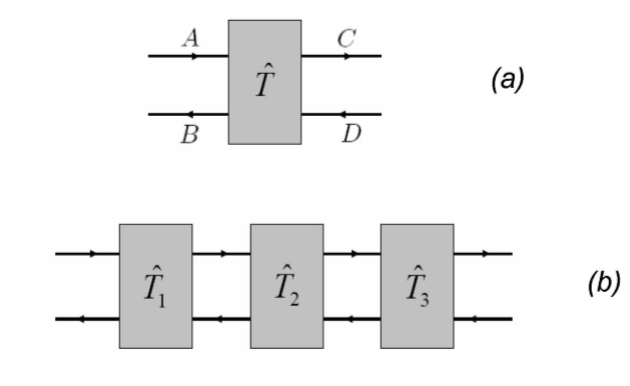
But this equation is not self-consistent, and cannot be true, for all b at least.

**3. Modified Shapiro approach**

So Shapiro’s results had two defects. First, it didn’t get the metallic regime right, and second, it was a little off at the critical point. He says that these defects can be remedied by using the generating function scaling equation, rather than the more naïve model, and also by extending the 1D model a bit. Remarks that the probability distribution P(g) must depend on two parameters L/ξ and z/L, because a single block would depend on L/ξ, by the single parameter scaling theory, and the number of blocks, or well, the number along the horizontal: Lz/L. Now is that the case? I don’t know…does this reconcile with our stuff? Ultimately, yeah…f(L/ξ,Lz/L) is functionally equivalent to f(L/ξ, Lz/ξ) – see scaling theory in physical stuff.

**3.0 Generalization of 1D equation**

Next, he proposes the following generalization of the 1D equation?



The 1D transfer matrix can be written as:



and likewise he assumes the building block can be written as (and he says that β are small quantities).



Note that the white noise model would correspond to



So his does not reduce to whitenoise, actually, for any choice of parameters. So let’s work out the recursion relation for ρ.



So we have:



So then, using <ε2> = αdz, <sin2()> = A0, we have:



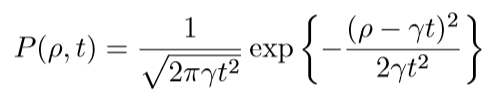
And therefore,

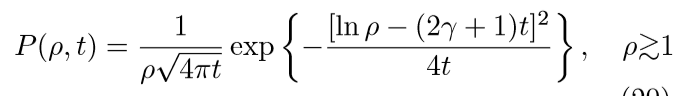


Note,



So observe that we’ve just introduced one extra parameter to be modeled somehow, and that is A0 – the angular average thingy. When A0 = ½, then γ = 0, and everything reduces to the usual 1D equation. Alternatively, I guess we can think of the fundamental parameter being γ, with A0 defined in terms of *it*. We’re reduced to the previous case if γ = 0, i.e. <sin2()> = ½, which would make sense itself. This also presupposes that it can be averaged independently of ρ, which he explicitly assumes. Take a look at his reference [29] which provides support for the presence of this γ term. Also check reference [30]. In any event, he gets the following results for small ρ and large ρ [remember this ρ is not Shapiro’s ρ],





Recall g = 1/(1+ρ), and for small ρ, we have g ~ 1 – ρ, and so the first corresponds to roughly normal distribution (I think this requires more support). And for large ρ, we have g ~ 1/ρ, and so the second roughly corresponds to ln-normal. But they are γ-dependent distributions. Does this make sense. Does this correspond to numerical simulations? For any finite γ, as t → 0, we get P0(ρ) = δ(ρ) which is what we should have.

Now he provides a different justification. So we pivot to the conductance. Again, he says that we need to interpret the Landauer formula



in the context of semi-transparent boundaries on either side, so that the ‘real’ definition:



would reduce to the former in the limit of small Ti. Maybe see [31] for more on this. So any real setup ought to have these boundaries that restrict T to small values. Says stuff I don’t understand about density of states and stuff. We start with:



We can develop a recursion relation for the λiphys as follows:



(do his subsequent result depend heavily on how this barrier T0 has been modelled?) and then increase L to L + ΔL, and he goes through arguments that for κ >> 1, which makes T small, and using a RPA, he gets the same equation as above, with the particular values:



This is consistent with his prior written relations between γ and A0. Perhaps universality will be achieved in this large κ limit (i.e. results independent of κ). *But note that when I use a white noise model, in conjunction with semi-permeable boundaries, I still get the usual DMPK back. So it seems that things are heavily dependenct upon his microscopic model.*

Also, if this γ term turns out to be relevant, then perhaps it emerges naturally from the GDMPK equation. He tries to generalize it explicitly get this term, but it might be that when you solve it for P(ρ) or whatever, that it just naturally comes out?

**3.1 Scaling of <G> assuming identical chains**

Now let’s revisit the scaling arguments undertaken in the last section. First we’ll consider a simplified approach. Note his ρ SDE is:



where dW is a white noise variable: <dW> = 0, <dW>2 = dt. And this results in the following PDE:



Then, we solves for the average resistance in terms of α (vice versa). So,



And then if we add in the classical part, we’d have:



and so we have:



and,



This is the same equations as before, and we’ll find that at the mobility edge: ρave = ε. Then as for the probability distribution:



Could also write this as:



Near the mobility edge, we get:



where,



This makes the probability distribution for g to be:



So this function has the following asymptotic behavior:



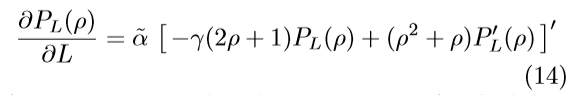
So only moments that will survive are those for which:



which seems to be basically, none of them, according to the top definition.

**3.2 Scaling of P(G) assuming independent chains**

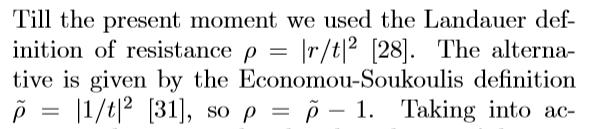
Anyway, let’s go to the moment generating function. So we’re going to accept:



i.e.,



with the replacement ρ → ρ0 + ρ, due to ambiguities in definition of conductance….



This will result in:



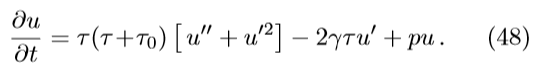
and we’ll get:



and  in his case.

**3.3 Consequences of P(G) scaling equation**

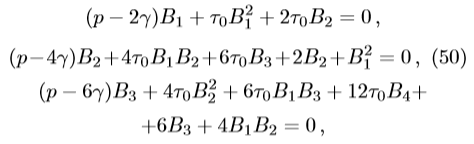
So we fill in F(τ) = exp(u(τ)), and write out PDE for u(τ)



Let’s check:



First consider the length independent solution to equation, corresponding to the critical point (or well, the long length limit of the critical point I guess). Makes series expansion solution in powers of τn



the coefficients Bn are the cummulants basically. And he says that in 2+ε dimensions, the cumulants get smaller so should be a good expansion.

**3.4 Reproducing results for d = 2+ε**

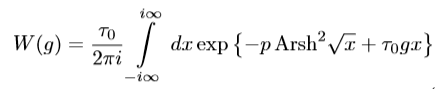
Can also make approximation Bn+1 >> Bn, valid for d = 2+ε, to facilitate recursive solving. With approximation: p >> γ >> 1, consistent with:



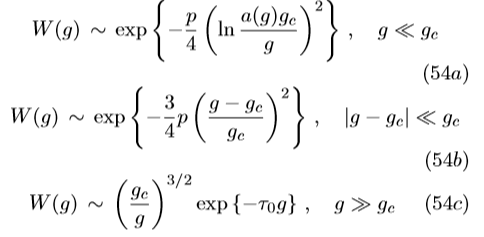
we have solutions: Bn ~ p/τ0n, to main order in p. This suggests a general solution u(τ) = pf(τ/τ0), which to main order in p, gives:



Then solving for P(g) via inverse Laplace transform we get:



Using SPA to calculate the integral (perhaps in limit of large p?), we get following results [gc = p/τ0, and a(g) is a logarithmic function of g]. And I guess we set τ0 = ε to match with expectations of what gc should be in 2+ε dimensions?



He says that extrapolating p back to unity or so, makes a(g) more or less constant over that specified range g << gc So then we’d have ln-normal for g << gc, proto-Gaussian for g close to the critical value, and then exponentially damped for g >> gc. Then consider length dependent deviations. We presume u = uc + eλLδu? And we get:

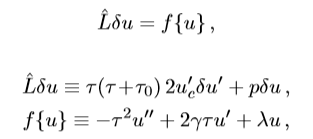


this follows from:



Then we say u(τ) = uc(τ) + δu? Let’s see…







Presuming a solution for δu ~ ΣBnτn, and solving to largest order in p, he gets for the cumulants <g>n (which are proportional to Bn):



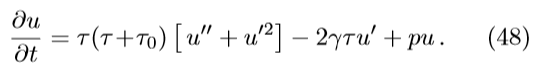
If we choose A ~ ε, 2γ+1 = 2/ε, then we reproduce the NLσM cumulant relation.

**3.5 Uniqueness of Solution**

Says some stuff. The initial condition u(0) seems to govern the type of solution we get. But we just choose the physically realistic one.

**3.6 Universal Tails**

If we solve



In the large τ (small g?), he says we get:



which justifies the ln-normal distribution contention for small g. Does it? The large g (small τ), solution is determined by u(τ) in the region of Reτ = -τ0, below which u(τ) picks up an imaginary part. Says we get:



which corresponds to:



which are the universal exponential tails he made reference to earlier in the paper.

**3.7 Phase Diagram**

So he says that for practically the entire p-γ plane, we have the following result:



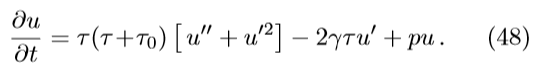
and also,



Constant values of <g> and <g2> correspond to straight lines in the p-γ plane. Constant values of both would seem to correspond to the critical point, no? Not sure how what substantial result is discussed in this section besides delineating the region of possible (p,γ)’s. Certainly we need -B1 and B2 to be positive, which would define a region of space.

**3.8 Metallic, Insulating Regimes**

Solving



if we do for large p ~ ξ/L (a positive definite parameter), and arbitrary γ (must be greater than -0.5), then we get a Gaussian distribution. But shouldn’t we get a ln-normal distribution for γ corresponding to z/L >> 1?

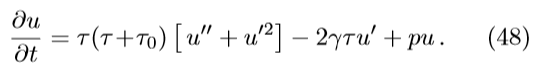
For small p ~ ξ/L, and arbitrary γ we get a ln-normal distribution.

This seems reasonable.

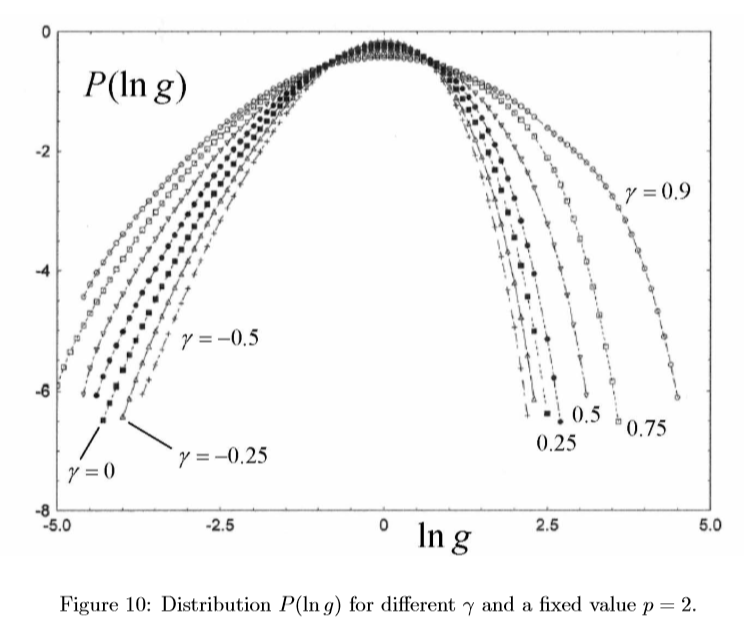
Again notes that p and γ are in correspdance with L/ξ and z/L. Mentions cubic conductors correspond to a certain line in the p-γ plane.

**3.9 Critical Region**

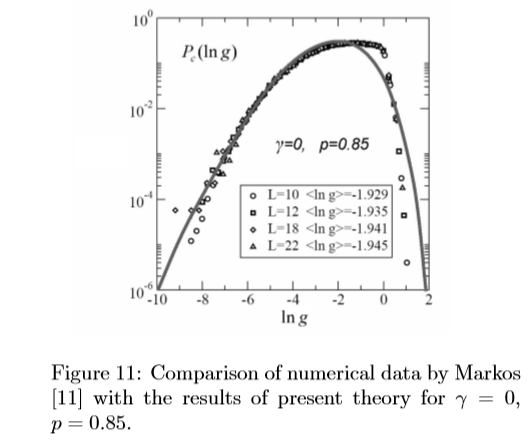
Going back to equation:



What initial condition do we use for u? Seems to be u(0) = 0. Recall p = f(L/ξ) is a positive definite parameter, and we want to solve this equation in the region p ~ 1. This evidentally complicates matters enough that we need to solve the ODE numerically, before performing the inverse Laplace transform. The result, for p = 2, and different values of γ is given below:



He finds that for large γ (> p/2 or so) the distribution is approximately ln-normal, while for small γ, close to one-sided ln-normal. He finds best agreement with 3D numerical results when γ = 0, p = 0.85:



Note that it shows a distribution that has quickly settled into its length-independent form. And also note that it isn’t power-tailed, unlike what we got with Shapiro’s results.

Changing τ0 just shifts the distribution left/right, but doesn’t affect its form. So I guess it’s like fixing the overall <g>?

He maintains that the discrepancy for lng > 0 is mainly due to the difference between Markos and his definitions of conductance. With his, there can be no singularity, unlike with Markos’s. And so it is no surprise that his results smear out the singularity. Can’t he use Markos definition of conductance and check for sure?

**4. Conclusion**

p and γ are functions of L/ξ and z/L, respectively. The exact correspondance depends on the ‘model’ used (perhaps analogous to microscopic model used?). The fact that we have a ξ at all seems to suggest we’re already the insulating side of things, though? How are p and γ in correspondance with L/ξ and z/L? p~ α/L so OK if α is related to ξ somehow. But isn’t γ an angular average thing?

For large p he finds Gaussian distribution consistent with metallic regime.

Does this mean small L/ξ and is z/L < 1 or so?

For small p he finds ln-normal distribution.

Does this mean large L/ξ and z/L >> 1 or so?

For p ~ 1, and γ within a certain space he finds one-sided ln-normal distribution.

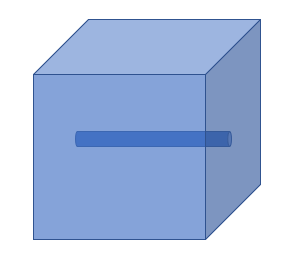
And then for a certain choice of parameters he finds he can reproduce the <g>n in d = 2+ε obtained from the NLσM. He says that extrapolation to d = 3 provides acceptable agreement for γ = 0, p = 0.85. But in this case p is our only d.o.f., and γ would be fixed?

He does not find any singularity in the distribution. Okay.

**Comments**

**2.1 Scaling of <G> assuming identical chains: my approach**

Let’s examine an alternate justification. Let’s presume that we have some arbitrary block.



And we’ll presume that at any length, a small change in scale results in classical change tranversely, and quantum change longitudinally. Is there a reason we may presume this holds true for small changes of scale? Or how is this a better assumption than assuming it holds at all scales, besides the fact that it wouldn’t be self-consistent as an RG scaling equation? Not sure. But then we’d get the following RG flow:



which is equivalent to Shapiro’s equation. Another way to say it is like this. We’ll just presume that we add an amount ΔL to the length and width. And so,



So our scaling equation would be:



and so,



And this doesn’t work, as we can see – it’s not even of the proper form as β(ρ) would depend on L. At this point, we could do what Shapiro does, and solve for α,



then we’d have:



which is of the proper form,



The rational for this seems to be that the coupling constant is presumed to depend on L itself, which makes sense, as σ ~ 1/A. Note that if we just assumed, more naively, that the classical formula holds for all transverse changes in scale, and the quantum for all longitudinal ones, then we’d get:



But this law wouldn’t be a valid scaling law (see that file) . And, wouldn’t evince any transition as far as I can tell. Now naively setting gL = 1/2ρL, we have?



and so,



Anyway, we can attempt to solve for g near the fixed point. We’d need to linearize the equation.



For large g we have:



Setting d-2 = ε, then we have:



Then we do a linear approximation in lng about the fixed point.



So in the vicinity, then, we have:



The ‘critical’ exponent is ε/2. I don’t know if that’s related to his quoted correlation length thingy 1/ε or not.

**2.2 Scaling of P(ρ) assuming identical chains: my approach**

Now we’ll reprise our argument. Assume that at any length L, the resistance scales classically-quantumly in the transverse/longitudinal direction. Let’s say it’s a random variable, ρN(L). Then according to quantum stuff, we have:



where dW is a white noise variable: <dW> = 0, <dW>2 = dt. And if we add in the classical part, we’d have:



and so we have:



Then it follows that:



and when we do the α replacement thing like above, we’ll get:



I do this differently in the Shapiro file.

**2.3 Scaling of P(g) assuming independent chains: my approach**

Let’s do a little more sophisticated approach. We’ll still presume that the individual wires scale as:



Changing variables to the conductance, defined as:



and recalling,



we’d have:



So then the PDE would be:



This matches his result. After 4 hours of adding the wrong two terms together. Suppose we know the probability distribution of the conductance of one of these chains. So what equation does PN(G) obey (where G = Σgi)? It’s best to look at it from the perspective of characteristic functions. What ODE does its characteristic function obey? Let’s define:



Then,



Another way we could’ve done it is, from formula in Appendix:



that:



Say you have N variables, and want to find the distribution of the sum of these variables? Let’s say:



Assuming they’re independent,



OK now let’s take the z derivative of each side:



Now the process begins in reverse. Problem is, we can differentiate w/r to G to bring down τ’s. But then we’ll have ∂/∂τ’s, and these we won’t be able to get rid off prima facie, unlike last time. Anyway, let’s go back to FN = F1N itself. If we can write a differential equation for it, and solve it, then we can take the inverse Laplace transform and get PN(G). So let’s say that for infinitesimal changes in length ΔL, FN(L) scales tranversely according to power (N+ΔN)/N = (L+ΔL)d-1/Ld-1 = (1+ΔL/L)d-1. And then that longitudinally, it scales in a 1D fashion, as found above. This scaling law makes more sense for the moment generating function, than it did for the random variable itself. Now let’s construct FN(τ,z), suppressing the τ argument,



And so we can conclude:



And so we get his result. Now, ostensibly, we should be able to get this equation from the N-channel DMPK equation itself, right?. And in fact we do have PN(**λ**) itself in all regimes, formally, and so we could get PN(G). So how do these compare? First, there is a qualitative difference in that the DMPK equation assumes isotropy, whereas, this would assume complete anisotropy.

**What about CLT though?**

Don’t think this analysis is relevant since the scaling law used is not absolute per seʹ, but incremental….maybe it is?

But consider the expected behavior of this equation from the CLT. The conductance of a given chain is <g(z)> ~ e-αz, for weak disorder I guess, and the conductance of the whole bunch of them is <g(z)> ~ Ld-1e-αz, therefore. But the conductance, being a sum of random independent variables, should converge to a *Gaussian* distribution in the limit of large L. But this runs into two problems: first, while this is good in the metallic limit, this violates the known log-*normal* Q1D distribution. And second, if we set z = L, then it is inevitable that the distribution scales towards an insulating state (ever decreasing <g>), and we do not capture the phase transition. Plus, it seems moments will always be calculable, and never diverge.

How do we avoid CLT? Let’s consider case where τ(z) is fixed. Then,



Then if we want to go back to the distribution we’d do:



**3.2 Scaling of P(G) assuming independent chains**

Anyway, let’s go to the moment generating function. We’ll reprise our earlier analysis.



Changing variables to the conductance, defined as:



and recalling,



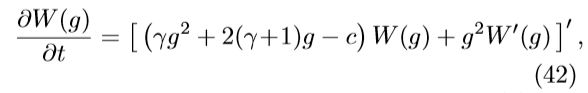
we’d have:



So then the PDE would be:



This bears resemblence to his result.



where,



But is missing my g3Pʹʹ term. Proceeding, we define:



Then its PDE is:



So we have:



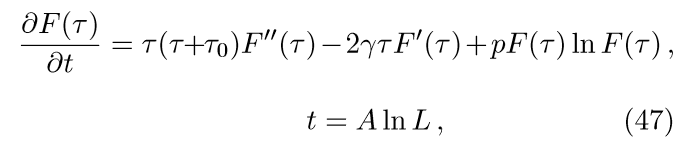
Then the total moment generating function would be given FN = F1N. And we develop its scaling equation the same way as before. We assume that it scales this way transversely, but via the above equation longitudinally.



Factoring out the α, we get:



This sort of but not really at all congrues with his result.



Could say τ0 = γ-1. And there is an overall factor of L missing. That might be a typo. But then where did the ∂3/∂τ3 term go – this stems from his leaving off the g3 term in the dP/dg equation? So he says we should’ve gone back to the original ρ-equation, and, using the alternative definition of conductance, ultimately made replacement ρ → ρ + ρ0 [i thought the present ρ equation with that γ term already took this into account?].

Other problem is the pre-factor, must be case that:

